

# Fractional quantum Hall effect in topological insulators: The role of Zeeman effect

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## Abstract

We study the role of Zeeman effect in fractional quantum Hall effect (FQHE) on the surface of topological insulators (TIs). We show that the effective pseudopotentials of the Coulomb interaction are reformed due to Zeeman effect, which are quite different from those in graphene. By exactly diagonalizing the many-body Hamiltonian in the sphere geometry, we find that the ground state energies and the excitation gaps at  $\nu=1/3$  FQHE between the  $n=\pm 1$  Landau levels (LLs) render asymmetry, and the FQHE state at the  $n=1$  LL is more robust than that at  $n=-1$  LL since the excitation gap at  $n=1$  LL is larger than that at  $n=-1$  LL.

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## I. INTRODUCTION

Because of the rich physical properties in fractional quantum Hall effect (FQHE), it has proved to be a vital and exciting subfield of condensed-matter physics to the present day. The electronic state giving rise to the experimentally [1–5] observed FQHE features should be of a many-particle origin, and has been studied intensively with various theoretical tools [6–16]. At present, most of experiments on this issue are performed in high-mobility GaAs-based semiconductor structures. However, it is still a challenge to detect the fractional quantum Hall state with high filling factors [17] (such as  $\nu=5/2$ ) due to the restrict of carrier mobility and the weak electron interactions in two dimensional semiconductor materials.

Fortunately, recently discovered two-dimensional electron systems, including graphene [18, 19], two-dimensional HgTe quantum wells [20, 21], and surface state of three-dimensional topological insulators (TI) [22, 23], which possess linear dispersion relation near the Dirac points with high Fermi velocity, may be good platforms to detect the FQHE. The unusual integer quantum Hall effect (IQHE) [24, 25] with the  $4e^2/h$  separation in the quantized Hall conductivity has been found experimentally in graphene, where the Landau levels (LLs) are four-fold degenerate without Zeeman effect and weak spin-orbit couplings. Soon after this discovery, Nomura and MacDonald [26] suggested the interaction-driven quantum Hall effects with intermediate integer value of  $e^2/h$  because of the charge gaps due to the electron-electron interaction. FQHE in graphene was also predicted theoretically [27–29], and the  $1/3$ -FQHE state was later measured [30, 31] successfully in suspended graphene.

Because of the Dirac nature of the electrons, many properties of the two-dimensional and three-dimensional TIs are similar to those of graphene. Recently, the graphene-like IQHE has been observed in strained bulk HgTe [32] and the LL spectrum has been measured in Bi<sub>2</sub>Se<sub>3</sub> by using scanning tunneling microscopy [33, 34]. Features in the Hall resistance at fractional filling factors have been speculated to be related to the FQHE of TIs [35, 36].

However, the finite thickness of TI sample in the growth direction will affect the nature of the FQHE states on TI surfaces [37], which is in contrast to the case in graphene. In addition, there exists another remarkable difference between graphene and TI, among which topologically protected Dirac cones on the surface of three-dimensional TI arise from the real spin-orbit coupling while the Dirac cone in graphene results from the pseudospin-orbit coupling. As a consequence, the external magnetic field induced Zeeman effect in TI and

graphene are quite different. The Zeeman effect reforms the LLs as  $sgn(n)\sqrt{\Delta_Z^2 + |n|(\hbar\omega)^2}$  in TI ( $\Delta_Z$  is the Zeeman energy,  $n$  is the LL index and  $\omega=eB/mc$  is the cyclotron frequency), while in graphene the LLs split as  $\pm\sqrt{|n|\hbar\omega}\pm\Delta_Z$  in the presence of Zeeman effect. Thereby, the influence of Zeeman effect on FQHE in TI should be quite different from that in graphene. Furthermore, on one side, the Zeeman term can perturb the Landau quantization in TI materials for a sufficiently large g-factor, which can reach  $\sim 30$  in recent experiment performed on the surface of  $\text{Bi}_2\text{Se}_3$  [35], in contrast, the g-factor in graphene is very small. On the other side, sufficiently strong magnetic field as large as hundreds of tesla, which can also bring on prominent Zeeman effect, is achievable in current experimental capabilities.

Therefore, because of its importance both from basic point of interest and to the analysis of unconventional properties of TI-based FQHE, in the present paper we address the role of Zeeman effect in FQHE of the massless Dirac electrons on TI surface by presenting an attempt at the theoretical evaluation of the effective pseudopotentials of the electron-electron interactions and the ground (excitation) state nature at  $1/3$ -FQHE between the  $n=\pm 1$  LLs problems. We find the following facts: (i) The effective pseudopotentials of the Coulomb interaction  $V_{\text{eff}}^{(n,m)}$  ( $m$  is the relative angular momentum of two electrons) are reformed due to the Zeeman term. At the first few relative angular momentum  $m$ , we have  $V_{\text{eff}}^{(|n|,m)} > V_{\text{eff}}^{(-|n|,m)}$  for any LL  $n \neq 0$ , while by increasing  $m$  to a moderate value, an inverse of the effective pseudopotentials happens  $V_{\text{eff}}^{(|n|,m)} < V_{\text{eff}}^{(-|n|,m)}$ . (ii) By taking into account the Zeeman effect and exactly diagonalizing the many-body Hamiltonian in the sphere geometry, the ground state energies and the excitation gaps at  $\nu=1/3$  FQHE between the  $n=\pm 1$  LLs exhibit asymmetry. (iii) Comparing with that at the  $n=-1$  LL, the FQHE state at the  $n=1$  LL is more robust because the excitation gap at  $n=1$  LL is larger than that in  $n=-1$  LL in the presence of the Zeeman energy. These findings obtained can not be observed in graphene within Zeeman effect, because the Zeeman energies in graphene just translate the LLs by  $\pm g\mu_B B$ , but induce no influence on the the Haldane's pseudopotential of electron-electron interactions.

## II. MODEL AND METHOD

We start from the free electron low-energy effective Hamiltonian of TI surface in the presence of a perpendicular magnetic field, which is given by

$$H = v_F \boldsymbol{\sigma} \cdot \boldsymbol{\Pi} + g\mu_B B \sigma_z, \quad (1)$$

where  $v_F$  is the Fermi velocity and  $\boldsymbol{\Pi} = \mathbf{p} + e\mathbf{A}/c$  is the two-dimensional canonical momentum. It is easy to obtain the eigenstates of Eq. (1), by choosing symmetric gauge  $\mathbf{A} = B(-y/2, x/2, 0)$  and introducing the ladder operators  $a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2l} - 2l\partial_z \right)$ ,  $a = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2l} + 2l\partial_z \right)$ , as

$$\Psi_{n,m}(r) = \begin{pmatrix} \alpha_n \phi_{|n|-1,m} \\ \beta_n \phi_{|n|,m} \end{pmatrix}, \quad (2)$$

where  $\phi_{nm}$  is the eigenstate of the 2D Hamiltonian with non-relativistic quadratic dispersion relation in  $n$ th LL with angular momentum  $m$ . Here,  $z = x + iy$ , and the magnetic length  $l = \sqrt{eB/\hbar c}$ , which is the length unit throughout this paper.  $\alpha_n = -i \text{sgn}(n) \frac{\cos \varphi_n}{\sqrt{2(1 - \text{sgn}(n) \sin \varphi_n)}}$  and  $\beta_n = \sqrt{\frac{(1 - \text{sgn}(n) \sin \varphi_n)}{2}}$  for  $n \neq 0$ , while  $\alpha_n = 0$  and  $\beta_n = 1$  for  $n = 0$  LL, which contain the information of Zeeman effect since  $\varphi_n = g\mu_B B / (\hbar v_F l^{-1} \sqrt{2|n|}) \equiv \varphi / \sqrt{2|n|}$ . The corresponding eigenenergies are given by  $E_{n \neq 0} = \text{sgn}(n) \sqrt{(g\mu_B B)^2 + 2|n| \hbar^2 v_F^2 l^{-2}}$ , and  $E_{n=0} = -g\mu_B B$ , which are different from LLs in graphene  $E_n^{\text{gra}} = \pm g\mu_B B \pm \sqrt{2|n|} \hbar v_F l^{-1}$  taken into account the Zeeman splitting in the effective Hamiltonian at the Dirac point  $K$  (or  $K'$ ) of graphene  $H_0^{\text{gra}} = v_F \boldsymbol{\sigma} \cdot \boldsymbol{\Pi} \otimes I_{2 \times 2} + I_{2 \times 2} \otimes g\mu_B \mathbf{B} \cdot \mathbf{s}$  with  $\boldsymbol{\sigma}$  the pseudospin and  $\mathbf{s}$  the real spin.

In order to investigate the properties of TI surface FQHE at partially occupied LLs with fractional filling factors, we should firstly consider Coulomb interaction  $V(\mathbf{r}) = e^2/\epsilon r$  between two electrons in  $n$ th LL with relative angular momentum  $m$ . It is expressed as the Haldane's pseudopotential

$$\begin{aligned} V_{\text{eff}}^{(n,m)} &= \langle \langle nm_1; nm_2 || V || nm_3; nm_4 \rangle \rangle \\ &= \frac{\cos^4 \varphi_n}{4(1 - \text{sgn}(n) \sin \varphi_n)^2} V_m^{(|n|-1)} + \frac{\cos^2 \varphi_n}{2} V_m^{(|n|, |n|-1)} \\ &\quad + \frac{(1 - \text{sgn}(n) \sin \varphi_n)^2}{4} V_m^{(|n|)}, \end{aligned} \quad (3)$$

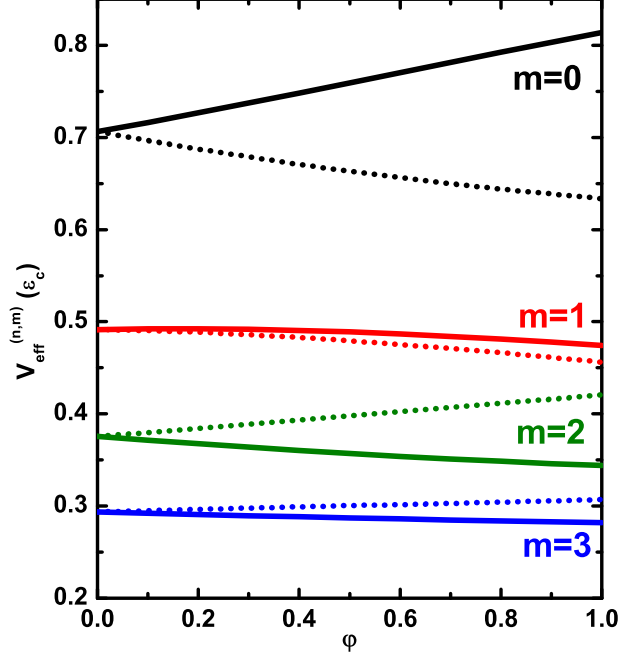


FIG. 1: (Color online) The effective pseudopotentials of the Coulomb interaction  $V_{\text{eff}}^{(n,m)}$  between two electrons at  $n=\pm 1$  LLs with the lowest four relative angular momentum  $m$  as a function of Zeeman parameter  $\varphi=g\mu_B B/\hbar v_F l^{-1}$ . The solid and dotted lines correspond to  $n=1$  and  $n=-1$  LLs, respectively.

where  $||nm_1; nm_1\rangle\rangle = \Psi_{n,m_1} \otimes \Psi_{n,m_2}$  and

$$V_m^{(|n|)} = \int \frac{d^2k}{(2\pi)^2} \frac{2\pi}{k} [L_{|n|}(k^2/2)]^2 L_m(k^2) e^{-k^2}, \quad (4)$$

$$V_m^{(|n|,|n|-1)} = \int \frac{d^2k}{(2\pi)^2} \frac{2\pi}{k} L_{|n|}(k^2/2) \times L_{|n|-1}(k^2/2) L_m(k^2) e^{-k^2}, \quad (5)$$

with  $L_m(x)$  the Laguerre polynomials. In what follows, the energy unit is chosen as  $\varepsilon_c = e^2/\epsilon l$ , and focus on the role of Zeeman effect in TI surface FQHE. It is clear that the Zeeman term can perturb the Landau quantization itself for a sufficiently large g-factor or a sufficiently strong external magnetic field  $B$ .

### III. RESULTS AND DISCUSSIONS

One can easily find from Eq. (3) that the effective pseudopotential at the  $n=0$  LL is independent on the Zeeman energy. The effective pseudopotentials at other LLs ( $n \neq 0$ ), however, are dependent on the Zeeman energy. Because the stability of FQHE phenomenon can only occur at  $n=0, \pm 1$  LLs on a TI surface [38], we only plot in Fig. 1 the effective pseudopotentials of the Coulomb interaction  $V_{\text{eff}}^{(n,m)}$  between two electrons at  $n=\pm 1$  LLs with the lowest four relative angular momentum  $m=0, \dots, 3$  as a function of Zeeman parameter  $\varphi$ . We notice that on one side, the pseudopotentials  $V_{\text{eff}}^{(1,m)}$  and  $V_{\text{eff}}^{(-1,m)}$  are degenerate in the absence of Zeeman term ( $\varphi=0$ ), which could also be straightforwardly observed from Eq. (3) by setting  $\varphi_n=0$ . On the other side, the degeneracy between  $V_{\text{eff}}^{(1,m)}$  and  $V_{\text{eff}}^{(-1,m)}$  is removed with increasing the Zeeman parameter  $\varphi$ . Interestingly, one can see in Fig. 1 that under the same Zeeman parameter  $\varphi$  (for example,  $\varphi=0.2$ ) the splitting of pseudopotentials for relative angular momentum  $m=1$ , i.e.,  $|V_{\text{eff}}^{(1,1)} - V_{\text{eff}}^{(-1,1)}|$  is smaller than that for other lower relative angular momentum  $m=0, 2$ , and 3. We also find in Fig. 1 that, for the electrons at LL  $n=\pm 1$  the pseudopotentials  $V_{\text{eff}}^{(1,m)} > V_{\text{eff}}^{(-1,m)}$  only for relative angular momentum  $m=0$  and 1, but  $V_{\text{eff}}^{(1,m)} < V_{\text{eff}}^{(-1,m)}$  for higher angular momentum  $m$ .

These results could also be seen from Fig. 2(a), where the red squares (blue triangles) correspond to  $n=1$  ( $n=-1$ ) with  $\varphi=0.2$ . In fact, it is a truth that at the first few relative angular momentum  $m$ , we have  $V_{\text{eff}}^{(|n|,m)} > V_{\text{eff}}^{(-|n|,m)}$  for any LL  $n \neq 0$ , while by increasing  $m$  to a moderate value, an inverse of the effective pseudopotentials happens  $V_{\text{eff}}^{(|n|,m)} < V_{\text{eff}}^{(-|n|,m)}$ . As an another example, we also show the pseudopotentials at LL  $n=\pm 2$  verse relative angular momentum  $m$  in Fig. 2(b).

Based on the pseudopotentials for Dirac fermions on the surface of TIs, one can further obtain the energy spectra of the many-body states at fractional fillings of the LL by numerically diagonalizing the many-body Hamiltonian in the spherical geometry [8]. In this point of view, thus the external magnetic field perpendicular to the surface of sample is equivalent to a fictitious radial magnetic field produced by a magnetic monopole at the center of sphere of radius  $R$ , which is related to the magnetic fluxes  $2S$  through the sphere. It is easy to demonstrate that in the units of flux quanta  $R=\sqrt{Sl}$ , and the many-body states could be described by the total angular momentum  $L$  and its  $z$  component  $L_z$ , while the energy just depends on  $L$ . In what follows, we investigate the system with the fractional filling factor

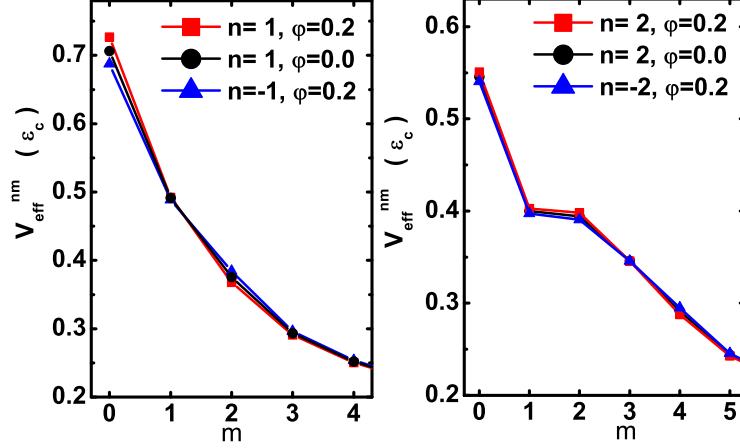


FIG. 2: (Color online) The effective pseudopotentials  $V_{\text{eff}}^{(n,m)}$  between two electrons at (a)  $n=\pm 1$  and (b)  $n=\pm 2$  LLs as functions of the relative angular momentum  $m$  with different Zeeman parameters  $\varphi=0$  and  $\varphi=0.2$ . The lines are a guide to the eye.

$\nu=1/(2p+1)$ , where  $p$  is an integer, especially the  $p=1$  case, i.e., the  $\nu=1/3$  FQHE state which is realized at  $S=\frac{3}{2}(N-1)$  in the spherical geometry, where  $N$  is the electron number.

The typical results of exactly diagonalized energy spectra per electron for the  $\nu=1/3$ -FQHE state at  $n=\pm 1$  LLs with the same Zeeman parameter  $\varphi=0.2$  as well as the degenerate energy spectra for the  $\nu=1/3$ -FQHE state at  $n=\pm 1$  without Zeeman term are shown in Fig. 3. One can clearly observe in Fig. 3 that the degenerate energy spectra (black) at  $\varphi=0$  split into two-class alike energy spectra when the Zeeman term are introduced. The excited energy spectra (red) at  $n=1$  LL are promoted while those (blue) at  $n=-1$  LL are degraded. From Fig. 3 one can also find two prominent features brought by the Zeeman term as follows: (i) The FQHE state at the  $n=1$  LL is more robust since the excitation gap at  $n=1$  LL is larger than that at  $n=-1$  LL in the presence of the Zeeman term; (ii) However, the promotion (degradation) of each excited energy is not same with each other, which indicates that the Zeeman term leads to the asymmetry of the LLs. These two properties of the Zeeman effect on the FQHE in TIs are unique, which can not be observed in the popular studied graphene system because the Zeeman energy in graphene Hamiltonian  $H_0^{\text{gra}}$  does not affect the Haldane's pseudopotential but just translate the LLs by  $\pm g\mu_B B$ .

We now turn to study the dependence of the excited gap on the Zeeman parameter  $\varphi$ . In Fig. 4 we show the excited energy gap (in unit of  $\epsilon_c$ ) versus Zeeman parameter  $\varphi$  for  $N=6$

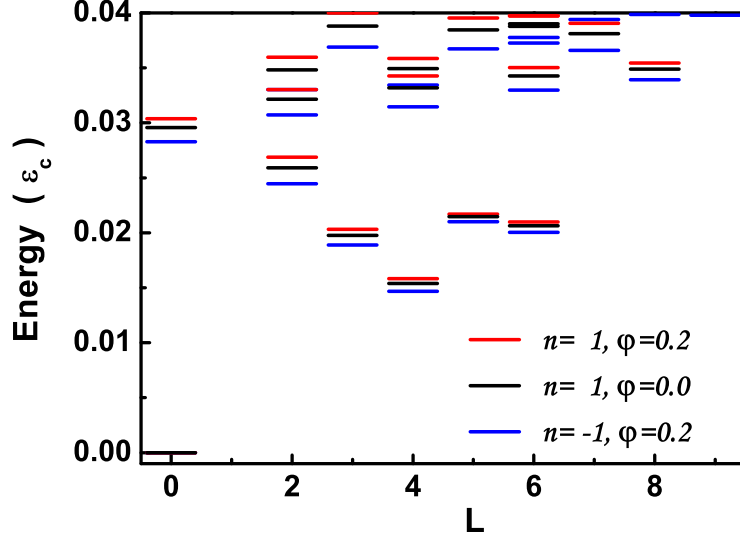


FIG. 3: (Color online) Exact energy per electron versus the angular momentum  $L$  for  $N=6$  electrons at  $\nu=1/3$  FQHE state. The blue and red lines correspond to the  $n=1$  and  $n=-1$  LLs, respectively. The Zeeman parameter is chosen as  $\varphi=0.2$ .

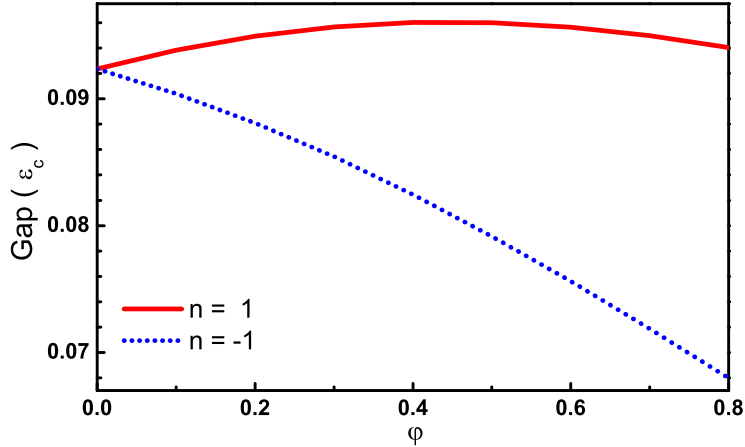


FIG. 4: (Color online) Excited energy gap versus Zeeman parameter  $\varphi$  for  $N=6$  electrons at  $\nu=1/3$  FQHE state. The red and blue lines correspond to the  $n=1$  and  $n=-1$  LLs, respectively.

electrons at  $1/3$ -FQHE state, whose corresponding lowest excited state angular momentum is  $L=4$ . One can clearly see that the excited gap for  $n=1$  LL is always larger than that for  $n=-1$  LL, which is consistent with the results in Fig. 3. Besides, the excited gap for  $n=-1$  LL monotonically decreases with increasing the Zeeman parameter (see the blue line in Fig.



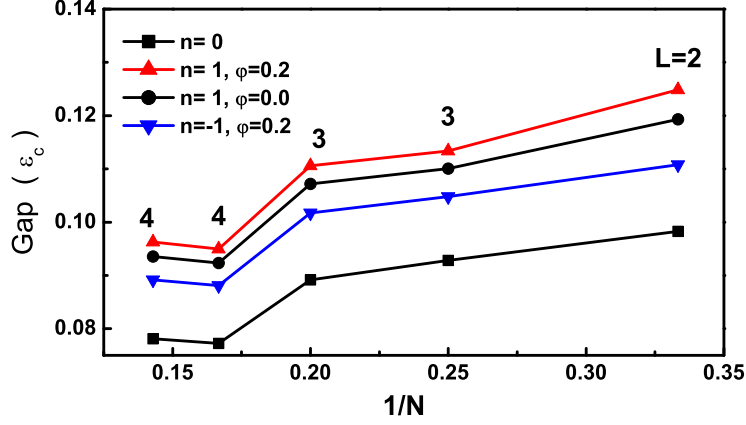


FIG. 5: (Color online) Excited energy gap versus the inverse of the electron number  $N$  for the  $n=\pm 1$  LLs with different Zeeman parameters  $\varphi=0.2$  and  $\varphi=0$ . The integers denote the angular momentums  $L$  of the lowest excited energy states.

4). Whereas, the variance of the excited gap for  $n=1$  LL (the red line in Fig. 4) is weak with increasing the Zeeman parameter, and it gradually increases to a maximum at  $\varphi \approx 0.42$  and then tends to decrease.

The above discussions focus on the system of  $N=6$  electrons, but the properties obtained herein are conserved when the electron number  $N$  is changed. The results of the excited gap as a function of  $N$  are shown in Fig. 5. We should point out that, firstly, for the  $n=0$  LL, the Zeeman term does not affect the FQHE in TIs and therefore the excited gaps are same for different values of  $\varphi$ . Secondly, the case of  $n=0$  LL is quite different from that for  $n \neq 0$  LLs. From Fig. 5 we see that the excited gap is promoted (degraded) for  $n=1$  ( $n=-1$ ) LL due to the Zeeman effect. Thirdly, the angular momentum  $L$  of lowest excited state alters with the electron number  $N$ , which is indicated by the integers in Fig. 5. For instance, the angular momentum of the lowest excited states for systems of  $N=4$  and 5 electrons are  $L=3$ , while this becomes  $L=4$  for systems of  $N=6$  and 7 electrons. We also studied the FQHE with the fractional filling factor  $\nu=1/(2p+1)$ , where  $p=2, 3, \dots$ , and obtain the same results as that with  $\nu=1/3$ .

#### IV. SUMMARY

In summary, we have studied the Zeeman effect on FQHE in TIs. Compared to previously studied graphene system, the combination of the real spin-orbit interaction and the Zeeman term bring on unique physical properties in TIs. We have found that the effective pseudopotentials of the Coulomb interaction are reformed owing to the Zeeman effect. By exactly diagonalizing the many-body Hamiltonian in the sphere geometry, we also found that the ground state energies and the excitation gaps at  $\nu=1/3$  FQHE between the  $n=\pm 1$  Landau levels (LLs) exhibit asymmetry. Comparing with the  $n=-1$  LL, the FQHE state at the  $n=1$  LL is more robust since its excitation gap is larger. We hope our predictions can be confirmed in future experiments.

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